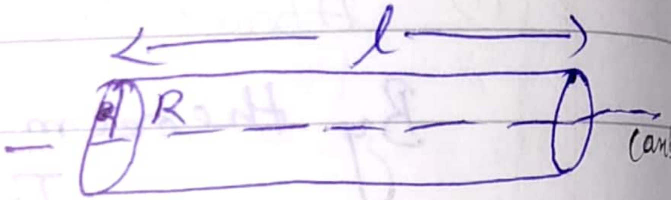
[5] Moment of Inertia of a solid cylinder

(i) About its own axis of cylindrical symmetry:



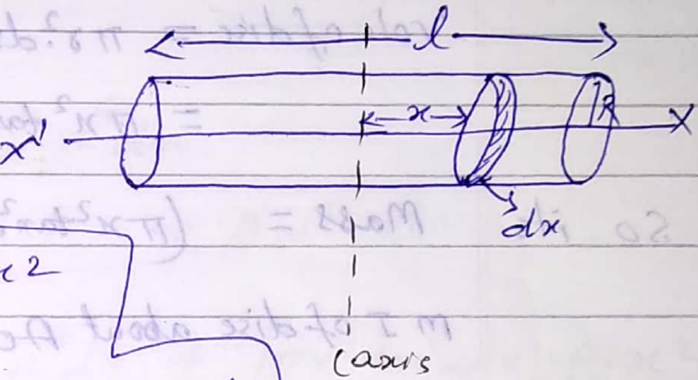
* The cylindrical may be considered as many number of small discs are jointed to form ~~cylindrical~~. cylinder or may be considered

a thick disc ~~whose~~ axis ~~axis~~ of cylinder maybe taken to passing through this thick disc and $\perp^{ar}$ to its plane. So M.I.

$$\boxed{I = \frac{MR^2}{2}}$$

(ii) About an axis passing through its centre and perpendicular to its axis of cylindrical symmetry.

$$\underset{\substack{\text{disc}\\(\text{about axis})}}{I} = \underset{\substack{\text{disc}\\(\text{about diameter})}}{I_{cm}} + M_{\text{disc}}\, x^2$$



$$I_{cm_{disc}} = \int \frac{M}{\pi R^2}(2\pi R)\,dx \cdot x^2$$

$$\because M_{disc} = \frac{M}{\pi R^2}(2\pi R)\,dx$$

Total ~~M.I~~ $I = 2\left[\int_0^{l/2} \frac{M}{R} \cdot 2x^2 dx + \frac{M}{R} \cdot 2dx \cdot x^2\right]$

$$\frac{4M}{R}\,l$$

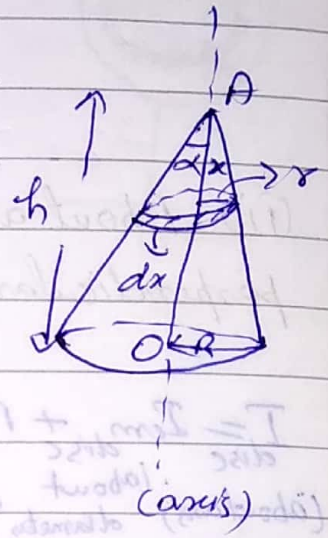
$$\underset{(\text{about diameter})}{I_{cm}} = \left(\frac{M}{l} \cdot dx\right)\frac{R^2}{4}$$

$$I_{dis} = \left(\frac{M}{l} \cdot dx\right)\frac{R^2}{4} + \frac{M}{l}dx \cdot x^2$$

$$I_{\text{Total System}} = 2\left[\int_0^{l/2}\left[\frac{M}{l}\frac{R^2}{4} + \frac{M}{l}x^2\right]dx\right]$$

$$\boxed{I = \frac{mR^2}{4} + \frac{ml^2}{12}}$$

(6) M.I. of a solid cone :-



(i) About vertical axis

radius of disc is r.

$r = x \tan\alpha$

Vol. of disc $= \pi r^2 \cdot dx$

$= \pi x^2 \tan^2\alpha \, dx$

So its Mass $= (\pi x^2 \tan^2\alpha \, dx) \cdot \rho$ (ρ = density)

M I of disc about AO. $= \text{mass} \times \frac{(\text{radius})^2}{2}$

$= (\pi x^2 \tan^2\alpha \, dx \, \rho) \frac{(x \tan\alpha)^2}{2}$

$= \left(\frac{\pi \rho \tan^4\alpha}{2}\right) x^4$

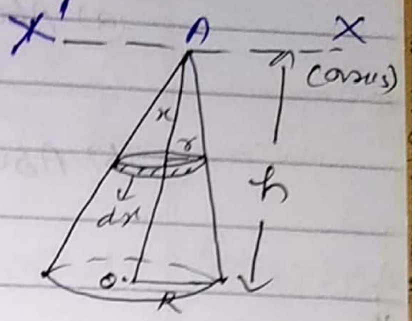
Now M.I. of entire cone about its vertical axis AO is given by

$$I = \int_0^h \left(\frac{\pi \rho \tan^4\alpha}{2}\right) x^4 \, dx = \frac{\pi \rho \tan^4\alpha}{2} \cdot \frac{h^5}{5}$$

$\therefore \frac{R}{h} = \tan\alpha$ and $\rho = \frac{M}{\frac{1}{3}\pi R^2 h} = \frac{3M}{\pi R^2 h}$

$$\therefore \boxed{I = \frac{3}{10} MR^2}$$

(ii) About axis passing through the vertex and ∥ to its base



M.I. of disc about its diameter

$$= \text{mass} \times \frac{r^2}{4}$$

$$= (\pi x^2 \tan^2\alpha \, dx) . \rho . \frac{r^2}{4}$$

$$= (\pi \rho \tan^2\alpha \, dx) x^2 . \frac{(x \tan\alpha)^2}{4}$$

$$= \frac{\pi \rho \tan^4\alpha}{4} x^4 dx$$

M I of disc about XAX' is

$$I_{disc} = \frac{\pi \rho \tan^4\alpha}{4} x^4 dx + (\pi x^2 \tan^2\alpha \, dx) . \rho . x^2$$

So M.I. of entire cone. $\{ I = I_{cm} + mr^2 \}$

$$I = \int_0^h \frac{\pi \rho \tan^4\alpha}{4} x^4 dx + \pi \rho \tan^2\alpha \, x^4 dx$$

$$= \frac{\pi \rho R^4}{4h^4} . \frac{h^5}{5} + \frac{\pi \rho R^2}{h^2} . \frac{h^5}{5}$$

$\{ \because \tan\alpha = \frac{R}{h}$

$\rho = \frac{3m}{\pi R^2 h}$

$$\boxed{I = \frac{3mR^2}{20} + \frac{3}{5} mh^2}$$

[7] (i) Moment of I. of spherical shell

$$I = \frac{2}{3} mR^2$$ (about any of its diameter)

(ii) M.I of a Solid sphere

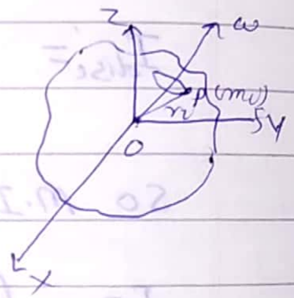
(a) About a diameter $= \frac{2}{5} mR^2$

(b) About a tangent $= \frac{2}{5} mR^2 + mR^2$

$$\boxed{I = \frac{7}{5} mR^2}$$

Angular Momentum and Inertia Tensor

$J = I\omega$



$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$I \rightarrow$ Inertia tensor

I is symmetric tensor

$$\begin{cases} I_{xx} = \sum_i m_i (r_i^2 - x_i^2) \\ I_{xx} = \sum_i m_i (y_i^2 + z_i^2) \end{cases} \rightarrow \text{moment of Inertia of body about X-axis}$$

$$I_{yy} = \sum_i m_i (r_i^2 - y_i^2) = \sum_i m_i (x_i^2 + z_i^2)$$

$$I_{zz} = \sum_i m_i (r_i^2 - z_i^2) = \sum_i m_i (x_i^2 + y_i^2)$$

$$I_{xy} = I_{yx} = -\sum_i m_i x_i y_i$$

$$I_{xz} = I_{zx} = -\sum_i m_i x_i z_i$$

$i \rightarrow m_1, m_2, m_3, m_4$ — corresponding position

$$I_{yz} = -\sum_i m_i y_i z_i = I_{zy}$$

Note:- Incase of a continuous, body, the summation sign is replaced by mass or volume Integration.

$$I_{xx} = \int (r^2 - x^2)\,dm = \int \rho(r)(r^2 - x^2)\,dV = \int \rho(r)(y^2+z^2)\,dV$$

$$I_{xy} = I_{yx} = -\int xy\,dm = \int -\rho(r)\,xy\,dV$$

and so on.

$$dV \equiv dx\,dy\,dz \quad \text{and} \int \equiv \int_x \int_y \int_z \text{ etc.}$$

$I_{xx}, I_{yy}, I_{zz} \longrightarrow$ Moment of Inertia about x, y & z axis's respectively

$$\underbrace{I_{xy},\ I_{yx},\ I_{xz},\ I_{zx},\ I_{yz},\ I_{zy}}_{\text{Called Products of inertia}} \rightarrow$$